Chapter 2
Polycentric models

The model we developed in the last chapter seems pretty useful and delivers predictions that are quite plausible. However, if you remember figure 1.1 at the beginning of the chapter, not all cities are monocentric. In fact, Bogotá and Mexico City do not look monocentric at all. Bogotá seems to have two centers, while Mexico City is pretty dispersed in space.

Putting two centers in the model of the previous chapter is not difficult, though it would be more tedious. But it would not address a more fundamental question. Where do these centers come from? Why would people want to form a city center and commute to it in the first place? We motivated the monocentric city with a city and a single port. But it would be hard to make these argument for cities that lack these places. We do see centers of economic activity without a natural advantage in the place where the center is located.

Maybe people just form a city center anywhere, because being together increases productivity. We call these **agglomeration forces**. They may be because of natural advantages in a place, but may be due to other factors. Maybe there are increasing returns to scale when firms and people agglomerate. Maybe markets work better when together. Maybe information flows more easily. We are going to be agnostic about how these agglomeration forces occur for now, and take them for granted with a reduced form.

Another issue with the monocentric model is that the location of firms is exogenous: All firms are in the center of the city. In reality, although firms tend to be in the city center, they also spread out around the city, forming clusters. We will endogenize the location of firms to see where they locate.

2.1 Fujita and Ogawa's polycentric model

Fujita and Ogawa (1982) develop a model of endogenous consumer and firm location under agglomeration forces. Compared to the simplicity of the monocentric model, these models with endogenous centers have several technical complications such as multiplicity of equilibria and no analytical solutions. We will follow the original presentation with a graphical approach. If you want to see analytical solutions, Fujita and Thisse (2013) and Duranton and Puga (2015) have some simplified, more tractable models.

2.1.1 Households

We'll place a closed city in a "ribbon" of land. Household locations are indexed by x, and in each location there is a unit of land. There is a continuum N of households, each of which consumes s_h units of land and z units of a composite good. Utility is $U(z, s_h)$ with the standard conditions. Each household works in a location x_w supplying one unit of labor, and receives a wage $w(x_w)$ from the firm that produces there. The budget constraint is:

$$w(x_w) = R(x)s_h + p_z z + td(x, x_w) \tag{2.1}$$

where $R(x)$ is land rent, p_z is the price of the good, t is a linear commuting cost and $d(x, x_w)$ is the commuting distance.

Now we start cheating to make things easier. Fix the lot size s_h so the problem becomes to choose the locations that maximize consumption z. Using the budget constraint, the problem becomes:

$$max_{x,x_w}\ z = \frac{1}{p_z}\left[w(x_w) - R(x)s_h - td(x, x_w)\right] \tag{2.2}$$

2.1.2 Firms

There are M identical firms that sell a single good at price p_o. Firm entry is allowed. You'll notice that this is not p_z: we are assuming that these goods are traded outside of the city. This is somewhat similar to the absentee landlords assumption in the previous chapter: we do not want to worry about general equilibrium income effects. There is free entry of firms to the city.

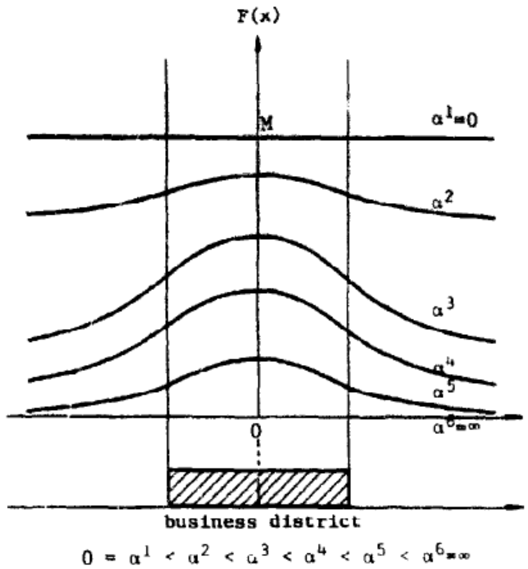
Firms produce with a Leontief production function: each unit of output is produced with L_b units of labor and S_b units of land. Because each firm uses L_b labor, the number of firms is $M = \frac{N}{L_b}$.

We introduce agglomeration in the model using a reduced-form *potential function* $F(x)$. Firms are going to be more productive when closer together, such that each location x is more productive if more firms are there, and this increase in productivity decays exponentially with distance.

$$F(x) = \int_{-\infty}^{\infty} b(y)e^{-\alpha d(x,y)}dy \tag{2.3}$$

Here α governs the decay of the potential function, $b(x)$ is the density of firms at x, and $d()$ is a distance function. Figure 2.1 shows a potential function for a given density pattern and different values of α.

Fig. 2.1 Potential function



Source: Fujita and Ogawa (1982)

We assume that this potential multiplies production, with a conversion factor β that translates potential into units of output. With these assumptions the profit function at location x is:

$$\Pi = p_0 mins_b, l_b \beta F(x) - R(x)S_b - w(x)L_b \tag{2.4}$$

The firms maximize this over x to choose an optimal location. Because of the Leontief assumption, output is fixed. p_0 and β are also fixed, so we can write the problem as

$$max_x \ \Pi = KF(x) - R(x)S_b - w(x)L_b \tag{2.5}$$

2.1.3 Equilibrium

There are 6 endogenous objects to be determined. 5 of these are familiar: firm density $b(x)$, household density $h(x)$, rents $R(x)$, wages $w(x)$ and the utility of the city

$\bar{u}$. The last object is a commuting pattern $P(x,x_w)$ that tells us how many households live at x and work at x_w. We'll outline the equilibrium conditions informally and turn them into formal statements one by one.

1. Firms and household optimize in their consumption and input decisions. We write their location decisions through bid-rent functions.
2. Households do not want to move. Formally, this means that wherever there is a household, rent must be equal to their bid-rent $Psi(x)$.

$$R(x) = \Psi(x,(\bar{u})) \text{ if } h(x) > 0 \tag{2.6}$$

   where $h(x)$ denotes the density of households.
3. Firms do not want to move. We can define bid-rent for firms in an an analogous fashion to consumer's bid-rent.

$$\Phi(x,\Pi) \equiv \frac{1}{S_b}\left[KF(x) - w(x)L_b - \Pi\right] \tag{2.7}$$

   Because of free entry, $\Pi = 0$ in equilibrium. The firm does not want to move if

$$R(x) = \Phi(x,0) \text{ if } b(x) > 0 \tag{2.8}$$

4. Land is exhausted at every point. Recall that 1 is the width of the ribbon city. For each x in the city:

$$s_b b(x) + s_h h(x) = 1 \tag{2.9}$$

5. Rent at each location is determined by the highest bidder

$$R(x) = max\{\Psi(x,\bar{u}), \Phi(x,0)\} \tag{2.10}$$

6. Outside of the city rent is equal to the agricultural rent, and land does not have to be exhausted.

$$R(x) = R_A \text{ if } x \notin [-\bar{x},\bar{x}] \tag{2.11}$$

$$S_h h(x) + S_b b(x) \leq 1 \text{ if } R(x) = R_A \tag{2.12}$$

7. Everything adds up:

$$\int h(x)dx = N \tag{2.13}$$

$$\int b(dy) = M \tag{2.14}$$

$$\int\int P(x,x_w)dxdx_w = M = N/L_b \tag{2.15}$$

2.1.4 Some equilibrium configurations

Solving for the above equilibrium turns out to be complicated. Lucas and Rossi-Hansberg (2002) have a complete description of how to solve a more general version of this model. Intuitively, you solve it by searching for fixed points in $h(x)$ and $b(x)$. Given a $b(x)$, (2.7) gives the firm's bid-rent function. Together with the rent, this will determine firm density. You can iterate this until convergence.

Instead of solving, we will just look at some equilibrium configurations. Figure 2.2 shows an equilibrium monocentric, symmetric configuration. This configuration has a potential function with large α and a large conversion β. This potential function is shown in panel (b). Panel (a) shows the equilibrium densities of households and firms. The center of the city from $-f_1$ to f_1 is commercial, and the rest of the city up to the fringes $-f_2$ and f_2 is residential. Panel (c) shows the wage gradient. This is the wage *paid* at location x. It has the slope of the (linear) commuting cost gradient and is higher at the center. Households are coming from the edge of the city, and must be compensated to travel a longer distance towards the center. Rents are also higher near the center (panel d) but fall a bit near the center because the firms right at the center must pay the highest wages but do not get an equivalent increase in potential. In fact the rent gradient looks like the difference between panels (b) and (c).

Fig. 2.2 An equilibrium monocentric configuration

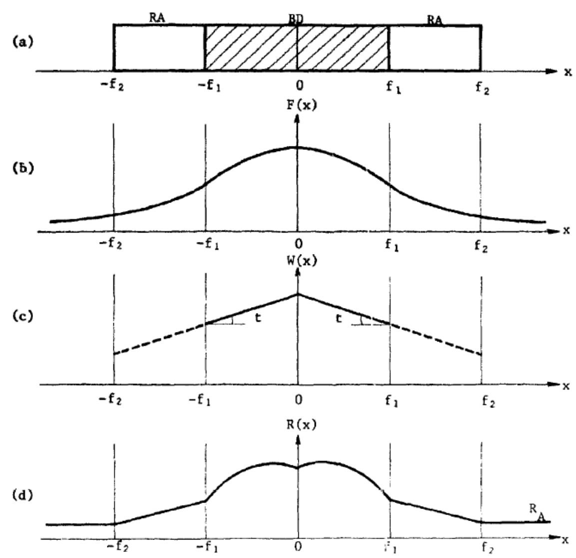


Fig. 2. Monocentric urban configuration.

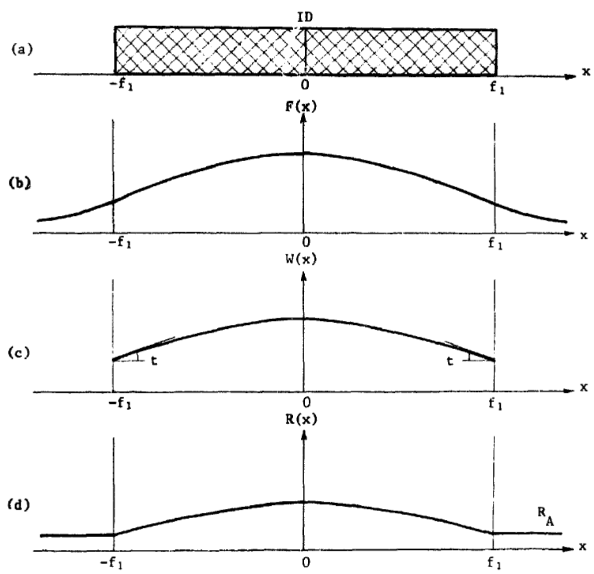
Source: Fujita and Ogawa (1982)

Also remember in panel (d) that in the rent gradient, we are looking a the maximum of the firms' and the households' bid rents. The firms' bid rent in the CBD must be higher than the households' to outbid them.

The relationship between the gradient of the potential function in panel (b) and the commuting cost gradient in panel (c) is important. Consider a flatter potential function, such that the slope of the potential function is smaller than the slope of the commuting cost gradient. In that case, if a firm moves towards the center, it has to pay extra wages to compensate the commuters, but the extra profits it would get are smaller that this extra cost. So firms would not want to be in the center, and the monocentric equilibrium breaks down. This same breakdown could happen if we increase commuting costs while keeping the potential function constant.

Figure 2.3 shows a mixed configuration without commuting. Firms and households coexist in $[-f1, f1]$. The wage gradient is flatter than the commuting cost gradient, because if it were steeper people would move to higher wages.

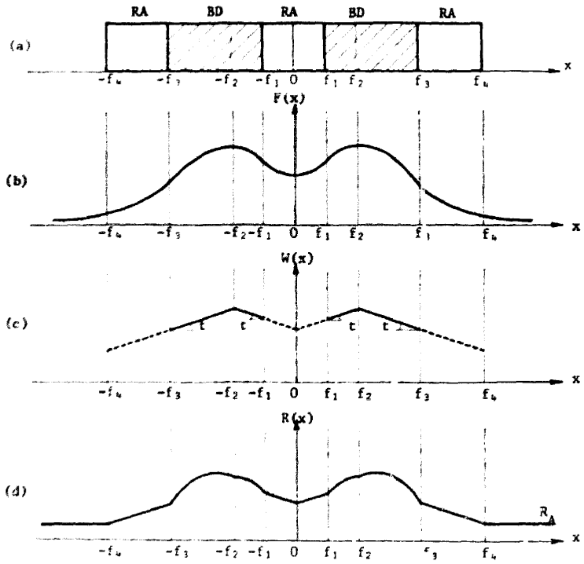
Fig. 2.3 An equilibrium mixed configuration



Source: Fujita and Ogawa (1982)

Figure 2.4 shows an equilibrium symmetric duocentric configuration. There are two business centers on either side of 0, around the places where the potential function peaks. Commuters come from either side of this business center. Wage gradients have the same slope as the commuting cost gradient. Commuters from the center split between the two CBDS. The rent gradient equals the wage gradient in the locations where households are located, but is steeper in the CBDs, because firms outbid households in these locations.

Fig. 2.4 An equilibrium mixed configuration



Source: Fujita and Ogawa (1982)

2.2 Summary

Fujita and Ogawa (1982) give an insight on how different city structures arise depending on the nature of commuting costs and agglomeration forces. From the structures we analyzed, it should be clear that this is the key tradeoff that determines where centers appear and where firms and households locate.

Lucas and Rossi-Hansberg (2002) has more details on these kinds of problems if you are a curious reader. In particular, they allow for a more general production function, such that output is not fixed. They also allows consumers to choose land size and consumption, so residential density varies more across the city. They also show conditions for uniqueness of equilibria, and develop an algorithm to compute equilibria.

We are still in a position where we take a reduced-form approach to these agglomeration forces. We would like to know where do they come from, instead of assuming some potential function. We will deal with that in the next chapter.