Chapter 1
The Monocentric Model

1.1 Motivation: Population and Land Use Gradients

We will motivate our benchmark model of cities by visualizing some patterns of how people locate inside cities across the world. Figure 1.1 shows a heatmap of population density for selected Latin American cities. On top, Lima, Perú displays high population density concentrated around a single point in the city. In the middle, Bogotá, Colombia, shows higher population density at two points in the city, but with overall densities lower than in Lima. At the bottom, in Ciudad de México, México, population is scattered around the entire city.

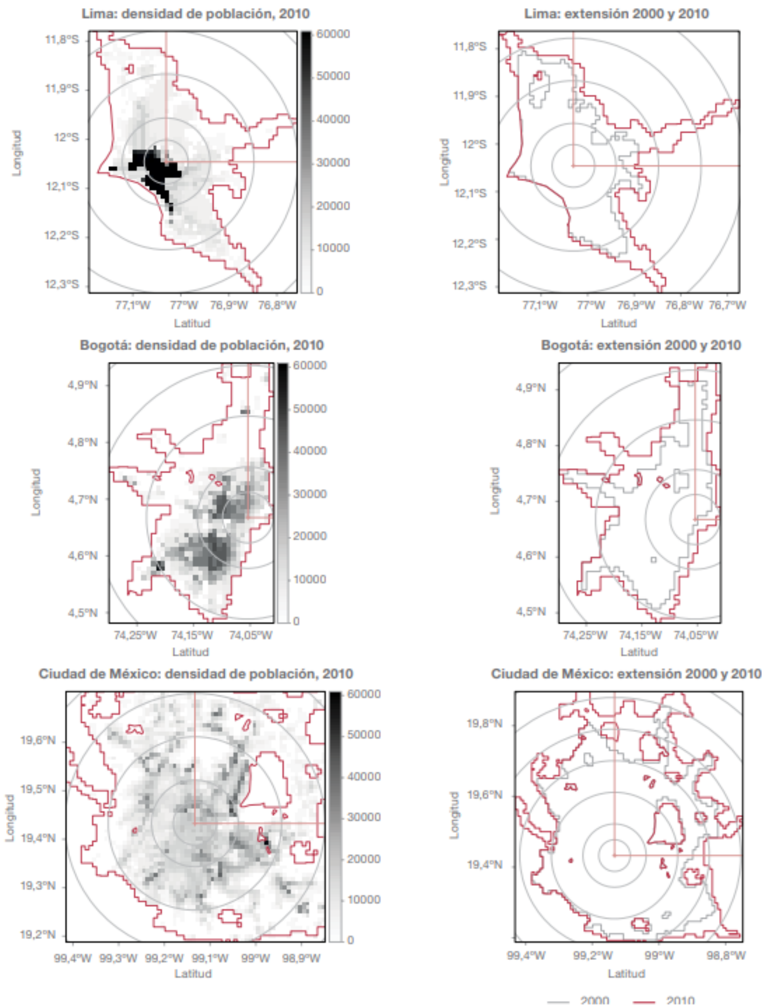
These different city configurations motivate our modelling of the cities. Lima displays high population density in a single point, which we will call the city center, or central business district. We will stat by modelling *monocentric* cities with a single center. We will take the center of the city as given, and ask, how does the population of the city distribute around this given center?

Monocentric cities were a natural starting point for modelling early cities, in which most business activity ocurred around a single point. Think, for example, about a coastal trade city with a single port. Most trade of goods and services would occur around the port, and residents would commute to the port daily. These patterns are still present in many cities today. In fact, a lot of business activity occurs around modern city centers, regardless of whether there is a port or not. Figure 1.2 shows how land is used according to distance from the city center. Most commercial use happens to locate around the city center. Multifamily dwellings with high density are also located near the center, while single-family houses with low density are located in the suburbs.

Another stylized fact, which will be familar to many of you, is that rents are higher in the city center. Figure 1.3 shows land prices by distance to the city center. As you can see, rents decrease as you are farther from the city center.

With these facts in mind, we will introduce a simple microeconomic model to account for these location patterns. Most of these patterns are going to be explained by the sheer power of the budget constraint.

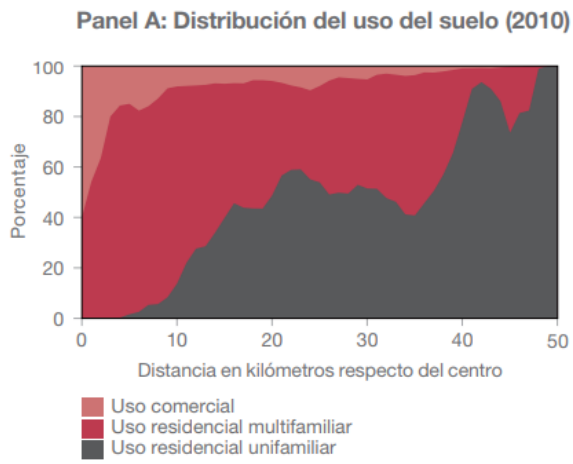
Fig. 1.1 Population density heatmap for selected Latin-American Cities



a/ Los gráficos de la izquierda sobre densidad de población identifican la densidad por kilómetro cuadrado para 2010 utilizando como fuente la población georreferenciada estimada por Landsat 8 (USGS - NASA, 2010). Los gráficos de la derecha identifican los límites de la extensión urbana para 2000 (en gris) y 2010 (en rojo). Para cada ciudad se tomaron como centros las siguientes referencias geográficas: Bogotá (Zona T); Lima (Plaza Mayor); Ciudad de México (Zócalo); Chicago (Cloud Gate); Madrid (Plaza Mayor) y París (Notre Dame). Tanto en los gráficos de densidad de población como en los de extensión, los círculos con centroide en el centro geográfico señalado anteriormente tienen radios que crecen de 5 km a 10 km, y después de 10 km en 10 km según la extensión de cada ciudad en el tiempo (es decir pasan de 10 km a 20 km, 30 km, 40 km y así sucesivamente). Por último, como referencia geográfica se agregan la latitud y la longitud de cada ciudad en grados.

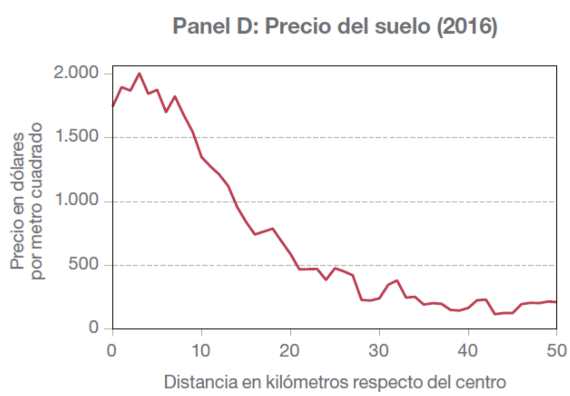
Source: Daude et al. (2017)

Fig. 1.2 Land Use in Buenos Aires, Argentina: 2010



Source: Daude et al. (2017)

Fig. 1.3 Land Prices in Buenos Aires, Argentina: 2010



Source: Daude et al. (2017)

1.2 The Monocentric City Model

This section follows Fujita (1989) and Brueckner (1987). As usual in economic models, we'll strip away many features of the problem in the beginning, which we'll reintroduce as the course goes along. Given many households that need to commute to a single central business district (CBD) for work, where will they locate? How much rent will be paid at each place in the city?

Here's the setup:

- There is a fixed continuum of *identical* worker households. This is the *closed city* assumption.
- The households are located on a featureless plane.
- We index locations by r. There is a single work location to which all workers commute to. This location is the central business district (CBD), denoted by r = 0. Think of a circle and travellers commuting to the center of the circle.
- There is a dense network of radial roads for commuting, e.g. you can take the straight line path to the CBD from any point.
- All households receive the same income Y.

1.2.1 The household's problem

The consumer's problem is:

$$max_{z,s,r}\ U(z,s)\ s.t.\ z+R(r)s=Y-T(r) \tag{1.1}$$

where z is consumption and s is the amount of space consumed, e.g., the area of your house. $R(r)$ is the endogenous rent at location r, and $T(r)$ is the exogenous commuting cost. We assume $T'(r)>0$.

To determine rent, note that the households are identical. So in equilibrium, nobody can be made better off by moving. This concept of spatial equilibrium will guide this model, and many others that follow:

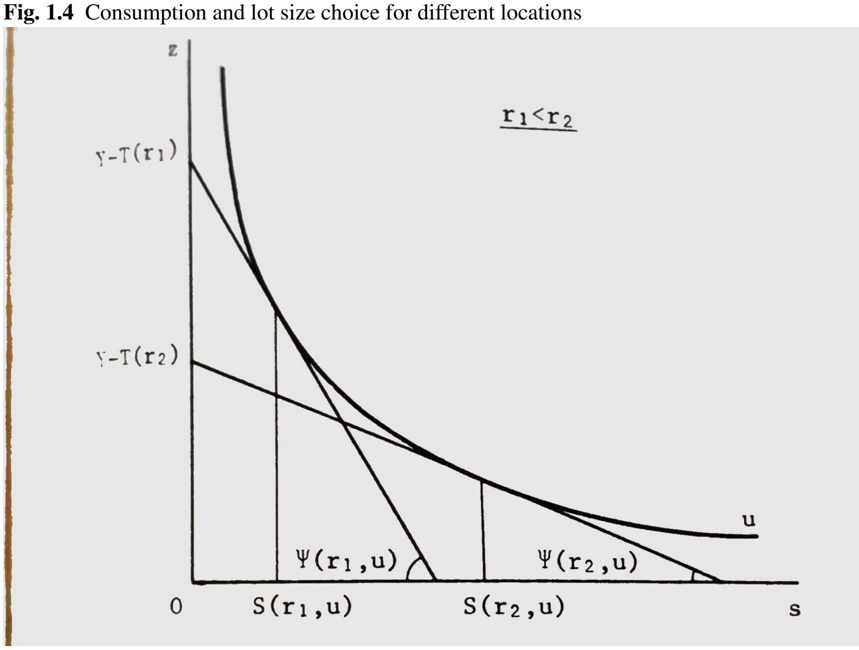
$$U(z,s)=\bar{U} \text{ for all } r \tag{1.2}$$

In practice, this means that all the households will attain the same indifference curve. The budget constraint will be given by their location. This is shown in Figure 1.4.

Now that we have fixed ideas, let's turn back to the optimization problem in (1.1). The FOC are:

$$\begin{aligned} U'_z &= \lambda \\ U'_s &= \lambda R(r) \\ 0 &= -\lambda T'(r)-\lambda R(r)s \end{aligned} \tag{1.3}$$

Fig. 1.4 Consumption and lot size choice for different locations



Source: Fujita (1989)

The last condition implies $T'(r) = -R'(r)s$. Since we assumed $T'(r) > 0$, this implies $R'(r) < 0$.

This implies that rents are lower as we move further away from the city center, and it is remarkable that we obtain this result with so little structure. Since commuting costs increase with distance, rent also increases with distance to equalize indirect utility across locations. In other words, *rents capitalize commuting costs*. If the household is further away from the center, commuting costs go up and consumption goes down. Because all households must achieve the same utility level, rent has to go down.

We have not yet seen what happens with lot size and density as we move away from the center. To do this, let's introduce some other analytical tools.

1.2.2 Marshallian demands, Hicksian demands and Bid-Rent functions

Consider a restricted version, conditional on location, of the problem (1.1).

$$max_{z,s}\ U(z,s)\ s.t.\ z+R(r)s = Y - T(r) \tag{1.4}$$

This is a standard consumer problem. In this case, the optimization will return a Marshallian demand $\hat{s}(R(r), Y-T(r))$, with the optimal lot size for a given income and rent. We can also solve the analogous expenditure minimization

$$min_{z,s}\ z+R(r)s\ s.t\ U(z,s) = u. \tag{1.5}$$

Which gives us a Hicksian demand $\tilde{s}_{r,u}$, the optimal lot size for given utility level and rent.

We now introduce a new function, unique to this kind of analysis: bid-rent and bid-max lot size. **Bid rent** is the maximum rent that a household is willing to pay to locate at r given a utility level u.

$$\begin{aligned} \Psi(r,u) &\equiv max_{z,s}\ \frac{Y-T(r)-z}{s}\ s.t.\ U(z,s)=u \\ &= max_s\ \frac{Y-T(r)-Z(s,u)}{s} \end{aligned} \tag{1.6}$$

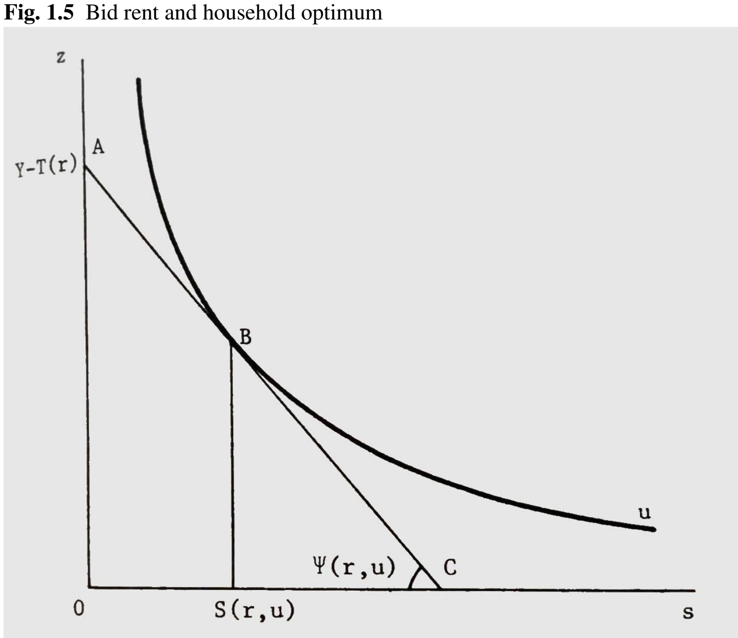
where we used the constraint to substitute for Z. The **bid-max lot size** is the lot size that the household acquires at location r when paying the bid-rent. We define in terms of the bid-rent and the Marshallian and Hicksian demand functions.

$$\begin{aligned} S(r,u) &\equiv \hat{s}(\Psi(r,u), Y-T(r)) \\ &= \tilde{s}(\Psi(r,u), u) \end{aligned} \tag{1.7}$$

This is useful for two reasons. First, it provides a convenient graphical approach to the problem. Bid-rent is the slope of the budget line for distance r that is tangent to the indifference curve u. There are several ways to show this. Notice that in the optimum for a given indifference curve, $\frac{U'_s}{U'_z}$ must equal rent, and that the rent that is paid to achieve this indifference curve must be the rent for the location that makes the budget line tangent to the indifference curve, which is exactly the bid-rent. This is shown in figure 1.5. The optimum lot size here is the bid-max lot size.

Bid-rent curves also allow us to translate this problem graphically from the (z,s) space to the (R,r) space. Bid-rent curves can be thought of as indifference curves in the (R,r) space, as they plot the maximum rent that a household would pay to live in each location r to achieve u. In equilibrium, this has to be the market rent, so equilibrium is where the bid-rent function is tangent to the market rent. This is shown in figure 1.6

Fig. 1.5 Bid rent and household optimum



Source: Fujita (1989)

The other reason why this is useful is because we can use bid-rent along with the Marshallian and Hicksian demand functions to derive comparative statics. For example, what happens with optimum lot size as we move away from the center?

Using the Marshallian demand, the derivative of bid-rent lot size w.r.t. location is

$$S_r' = \frac{\partial \hat{s}}{\partial r} = \frac{\partial \hat{S}}{\partial R}\left(-\frac{T_r'}{s}\right) + \frac{\partial S}{\partial I}\left(-T_r'\right) \tag{1.8}$$

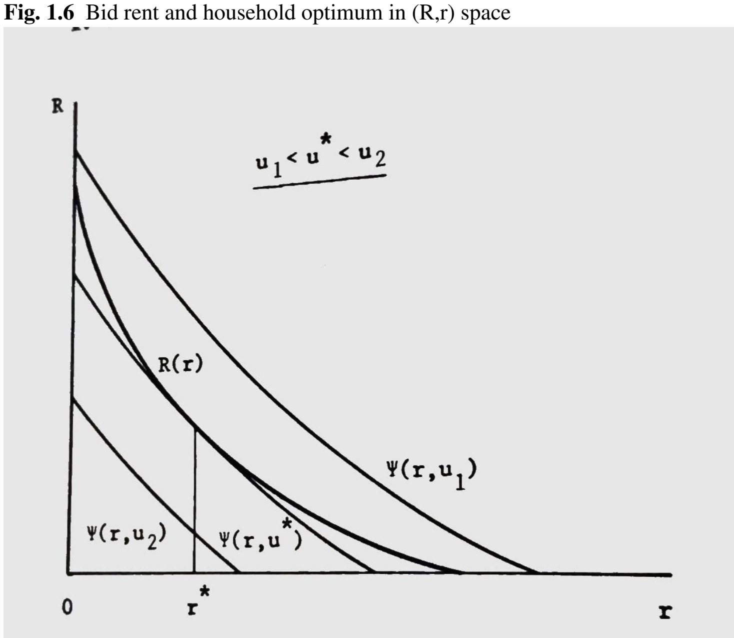
This looks ambiguous because of the usual income effects problem. But what if we use the Hicksian demand?

$$S_r' = \frac{\partial \tilde{s}}{\partial r} = \frac{\partial \tilde{s}}{\partial R}(-\frac{T_r'}{s}) + \frac{\partial \tilde{s}}{\partial u}\frac{\partial u}{\partial r} \tag{1.9}$$

This derivative is positive (why?). So there is no price effect ambiguity here. This derivative tells us the relationship between density and location, because density equals the reciprocal of lot size:

$$density = \frac{1}{s} = \frac{1}{space/person} = \frac{person}{space} \tag{1.10}$$

Fig. 1.6 Bid rent and household optimum in (R,r) space



Source: Fujita (1989)

So we have the two empirical observations that we saw at the beginning of this section. Now, let's see what additional testable implications we can derive from this theory.

1.2.3 Properties and comparative statics

There's many of these that you can check in the references, but we'll look at some interesting ones.

Bid-rent decreases with distance from the center, because the increase in commuting costs has to be compensated:

$$\Psi_r' = \frac{-T'(r)}{s} < 0 \tag{1.11}$$

Bid-rent is also convex w.r.t distance from the center

$$\Psi_{rr}'' = \frac{-T''(r)}{s(r,u)} + \frac{T'(r)}{s}\frac{\partial s(r,u)}{\partial r} \tag{1.12}$$

The first term is close to 0 if we assume commuting costs are approximately linear, and the second term is positive, so $\Psi''_{rr} > 0$. This happens because of how living further away is compensated. As the household moves further out, the rent must decrease to keep utility constant. But the household is also getting a larger lot size, so rent does not have to decrease as fast.

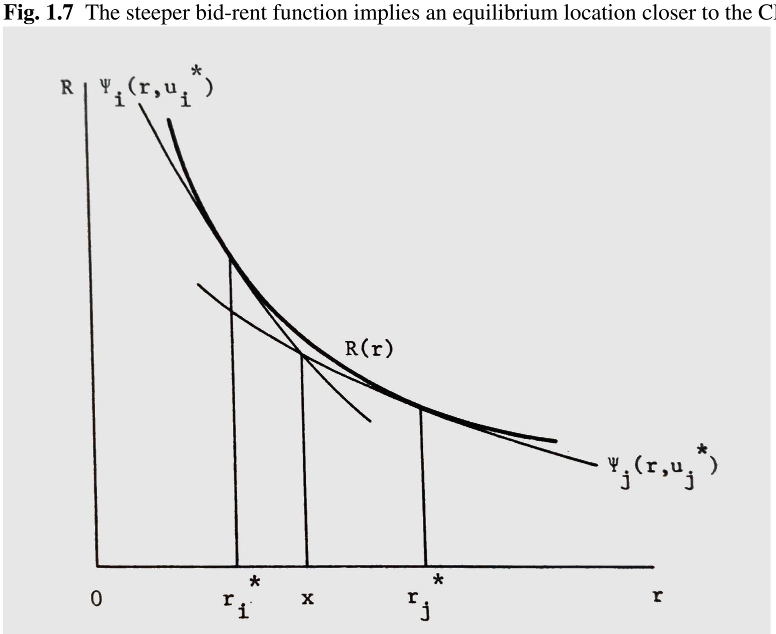
The bid-rent goes down as utility goes up:

$$\begin{aligned} \Psi'_u(r,u) &= \frac{\partial}{\partial u}\left[max_s \frac{Y - T(r) - z(s,u)}{s}\right] \\ &= -\frac{1}{s}\frac{\partial Z}{\partial u} < 0 \end{aligned} \tag{1.13}$$

This means that utility can only go up as rent goes down, since consumption must go up. When rent goes down, landlords bear all the utility costs.

Last, let's look at what happens with bid-rent as income rises. First, note that when two bid-rent functions cross, the steeper bid-rent function corresponds to a location closer to the CBD. See figure 1.7

Fig. 1.7 The steeper bid-rent function implies an equilibrium location closer to the CBD



Source: Fujita (1989)

Now, suppose that there are two households with different incomes $Y_1, Y_2, Y_2 > Y_1$. We would like to see which household's bid-rent curve is steeper when the curves cross. Assume that they cross in point $\bar{r}, \bar{R}$.

Assume that land is a normal good, such that:

$$\hat{s}(\bar{R}, Y_1 - T(\bar{r})) < \hat{s}(\bar{R}, Y_2 - T(\bar{r})) \tag{1.14}$$

Now recall that the slope of bid-rent is $\Psi_r' = -\frac{T'(r)}{S} = -\frac{T'(r)}{\hat{s}}$. This implies that

$$-\frac{\partial \Psi_1}{\partial r} = -\frac{T_1'(r)}{S_1} > -\frac{T_2'(r)}{S_2} = -\frac{\partial \Psi_2}{\partial r} \tag{1.15}$$

So the bid-rent function Ψ_1 is steeper, which means that the lower income household will live close to the CBD.

This merits some thought. Do you think this pattern holds in Mexico City? In the US it generally holds: richer households tend to live in the suburbs. But recently some cities have seen "urban revival" (Couture), and richer households are moving to the center. This may be due to amenities, which we are currently not looking at. But it may be due to transportation technologies and the opportunity cost of time. Fujita (1989) presents a time extended model we will not dwell into. Instead we'll look at an example:

LeRoy and Sonstelie (1983) show why time cost is important in the context of how cars spread across US cities. Think about two technologies: cars and buses. Cars have a lower time cost than buses, but they have a higher fixed and variable cost. Buses are cheaper but have a larger cost of time. The interaction of the cost of time and the cost of distance can generate interesting patterns.

- Without cars, everyone takes the bus and the rich live close to the center because they value their time more.
- With cars, the rich can now afford to live in the suburbs because it takes less time to commute in car and they can afford more space. So the rich live in the suburbs and the poor move to the center.
- But as cars become cheaper. the poor can afford them and move to the suburbs. Some rich move back to the center.

1.2.4 Closing the model

To close the model, we need to determine all the endogenous variables. These are the rent gradient $R(r)$, the lot size $s(r)$ and the utility level u. We also need to know where the city ends. Let's do that first.

Let's call the city boundary the urban fringe r_f. There are N households that need to live between the center and the fringe. Define $L(r)$ as the supply of space at each location. In a linear city, $L(r) = 1$. In a circular city, $L(r) = 2\pi r$. In equilibrium, everyone must find a place to live:

$$N = \int_0^{r_f} \frac{L(r')}{\hat{s}(R(r'), Y - T(r'))} dr' \tag{1.16}$$

Substitute the right hand side using the bid-max lot size, and you get a function r_f in terms of u only.

To determine r_f and u, we use the fact that the city is embedded in an agricultural area, where land rents are R_a. In the edge of the city, rent must equal this agricultural rent:

$$\Psi(Y - T(r_f), u) = R_a \tag{1.17}$$

The two previous equations determine r_f and u. To get the rent gradient, from (1.4), we get

$$R'(r) = -\frac{T_r'}{\hat{s}(R(r), Y - T(r))} \tag{1.18}$$

Which we can integrate to get

$$R(r) = R(0) - \int_0^{r} \frac{T_r'}{\hat{s}(R(r'), Y - T(r'))} dr' \tag{1.19}$$

This shows that the accumulated rent reductions as you move away from the center should be equal to the commuting costs increases per unit of space. This is another testable prediction. $R(0)$ should be $\Psi(Y - T(0), u)$.

Where does the money from rents go? There are two equilibrium concepts here. In the *closed city absentee landlords* model, the landlords do not live in the city, so the money just leaves the model. In the *closed city public land ownership* model, the land is collectively owned by the households, and the rents just go to them.

Another equilibrium concept is the **open city**. Here, we assume that households may enter and exit the city freely. Unlike in the closed city case, in the open city the level of utility achieved in the city is fixed and given by the agricultural rent. The household in the center must get the same utility than the households in the fringe, so

$$\Psi(Y - T(0), u) = R_A \tag{1.20}$$

this pins down u as a function of Y. Now to get the urban fringe, note that in the boundary rent must be equal to the agricultural rent, so

$$\Psi(Y - T(r_f), u)) = R_A \tag{1.21}$$

which determines r_f.

1.3 Welfare

If cities are such a great invention, we would like to see how much welfare they generate. Suppose you are a social planner that wants to build the least-cost city that provides utility u to every household out of N households. You have income $N\ Y_0$ and want to minimize the cost. Yo do so by choosing z and s for everyone. You don't have to choose both, since for each s households choose an optimum z. So you just have to choose how much space to give to everyone and where the city will end.

The cost of the city is the transportation cost, the cost of consumption and the opportunity cost of space, which is R_A, the agricultural rent that we would get if we did not build the city.

We will minimize cost (per unit of space), making sure that everyone fits:

$$max_{s(r),r_f}\ NY_0 - \int_0^{r_f} [T(r) + z(s(r),u) + R_A s(r)] \frac{L(r)}{s(r)} dr \tag{1.22}$$
$$s.t.\ N = \int_0^{r_f} \frac{L(r)}{s(r)} dr$$

The Lagrangian for this problem is:

$$\mathscr{L} = \int_0^{r_f} [Y_0 - T(r) - z(s(r),u) - R_A s(r)] \frac{L(r)}{s(r)} dr - \lambda \left[\int_0^{r_f} \frac{L(r)}{s(r)} dr - N \right] \tag{1.23}$$
$$= \int_0^{r_f} \left[\frac{1}{s(r)} (Y_0 - T(r) - z(s(r),u) - \lambda)) - R_A \right]$$

The last way of showing the problem shows two key features. First, the problem is equivalent to maximizing total net rents per unit of space, because (except for the Lagrange multiplier) what's inside the integral is bid-rent! So the total net rents are a measure of the welfare that the city generates. Fujita (1989) shows that the closed-city absentee landlords equilibrium is optimal, in the sense that this problem and the individual problem have the same optimality conditions.

1.4 Comparative statics

We'll now see what happens with the city as we change the parameters of the city. To do this, we introduce the boundary rent curve $\hat{(R)} = r$. This is the rent that we would get at the edge of a closed city, if the edge of the city were r. Recall the boundary condition:

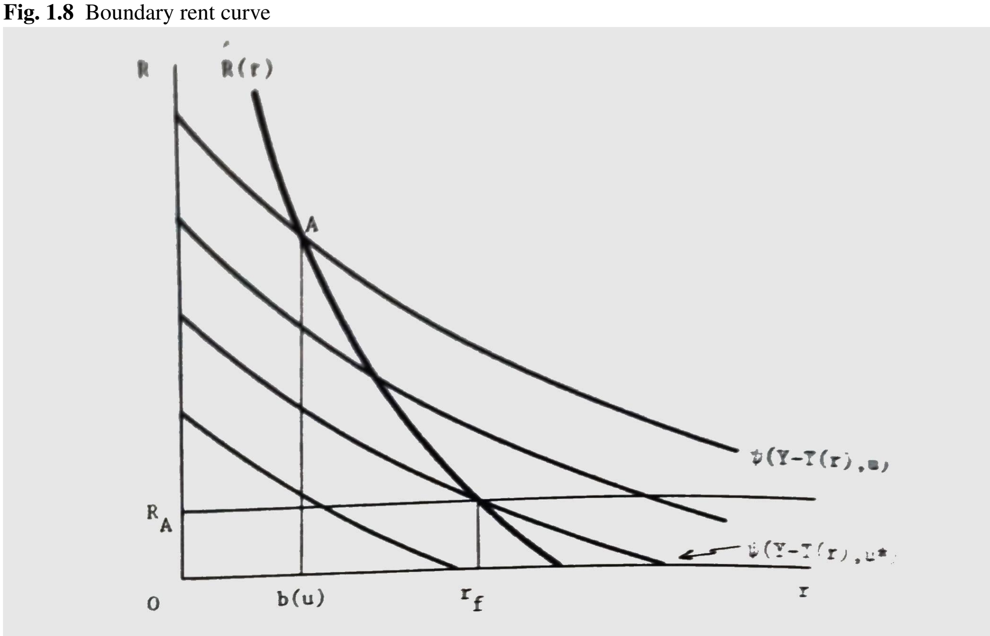
$$N = \int_0^r \frac{L(r')}{S(R,u} dr' \tag{1.24}$$

This defines a function $u(r)$, the utility that everyone will get if the edge of the city were r. The rent in this case will be just the bid-rent for that level of utility:

$$\hat{R}(r) \equiv \Psi(Y - T(r), u(r)) \tag{1.25}$$

This is a useful graphical device to do comparative statics, because it will show us how far the city will extend under changes in the parameters of the model. Note that at the edge of the city, $\hat{R}(r_f) = R_A$. This is shown in figure 1.8.

Fig. 1.8 Boundary rent curve



Source: Fujita (1989)

Now we are ready to use this graph for some comparative statics:

- When agricultural rent rises, utility goes down, the city becomes smaller, lot size decreases and and rents in the city increase.

- When population increases, $\hat{R}$ shifts out. Utility decreases, lot size decreases, rents and the urban fringe go up. This is intuitive, as there is larger competition for space.
- When income increases, the city expands and utility increases. Lot size and rents can go up or down depending on location and on the structure of transport costs and space supply.
- When transportation costs go up, lot sizes go down, the city contracts, and utility goes down. Rents can go up or down depending on location.

1.5 Housing supply

We'll now incorporate housing into the model to account for another stylized fact about cities: there are taller buildings in the center of the city. In fact, in the motivation we saw that multi-family housing is more common in the center of the city. This is not currently addressed because the supply of space is the same everywhere. This model is known as the Muth model.

We'll now assume that there are firms that produce housing with a CRS production function $F(s,k)$ using land s and capital k. The price of land is R, as before, and the price of capital is the interest rate i. $b = k/s$ is capital per unit of land, which can be thought of as building height. Firms charge a housing rent R_H for each unit of housing. Consumers choose lot sizes s_H.

Because of the CRS assumption, we can rewrite $F(s,k) = LF(1,k/s) = sf(b)$ with $f' > 0$ and $f'' < 0$. Profits are

$$\Pi = [R_H f(b) - R - ib]s \tag{1.26}$$

And the FOC are

$$\begin{aligned} R_H f'(b) &= i \\ R_H f(b) - ib &= R \end{aligned} \tag{1.27}$$

Consumers buy housing q at rent R_H. As before, from the consumer's side, R_H will decrease and lot sizes will increase as distance r increases.

How do firm's behave at each location? First, let's see how building height varies with location. From the first FOC

$$\begin{aligned} \frac{\partial R_H}{\partial r} f'(b) + R_H f''(b) \frac{\partial b}{\partial r} &= 0 \\ \frac{\partial b}{\partial r} &= -\frac{\partial R_H/\partial r \; f'(b)}{R_H f''(b)} < 0 \end{aligned} \tag{1.28}$$

So buildings are taller near the center. Land rents are lower away from the center. From the second FOC:

$$\frac{\partial R_H}{\partial r} f(b) + R_H f'(b) \frac{\partial b}{\partial r} = \frac{\partial R}{\partial r} \tag{1.29}$$
$$\frac{\partial R}{\partial r} = \frac{\partial R_H}{\partial r} f(b) < 0$$

This model also predicts that density is higher near the center, because building height is higher there. Notice that density $n(r)$ is

$$n(r) = \frac{people}{land} = \frac{1/land}{1/people} = \frac{Housing/land}{Housing/people} \tag{1.30}$$
$$= \frac{f(b)}{s_H}$$

So

$$\frac{\partial n}{\partial r} = f'(b) \frac{\partial b}{\partial x} \frac{1}{s_H} - f(b) s_H^{-2} \frac{\partial s_H}{\partial x} < 0 \tag{1.31}$$

1.6 Empirical predictions

We have covered a model that seems to explain some basic facts about cities reasonably well. Taking stock, it's also reasonable to think about what the model does not incorporate:

- Without housing, the model did not account for density. Density has been decreasing within cities over time.
- Land rent gradients have also been flattening over time.
- No rent regulation, or land zoning.
- No public land, or open space.
- No pollution, or congestion.

Now let's summarize some of the qualitative predictions of the model. First let's summarize the predictions for gradients within cities (Duranton and Puga, 2015). As we move away from the center :

1. Housing prices decline
2. Lot size increases
3. Land prices decline
4. Building height declines
5. Population density declines

There's also some city-level predictions, for the closed city case:

- When agricultural rent rises,, the city becomes smaller, lot size decreases and and rents increase.

- When population increases, lot size decreases, rents and the urban fringe go up.
- When income increases, the city expands
- When transportation costs go up, lot sizes go down and the city contracts.

The model also makes some quantitative predictions:

1. The rent gradient is the inverse of the commuting cost gradient, weighted by lot size consumption. This comes from the last condition in (1.4).
2. The ratio of the land price gradient to the housing price gradient should be equal to the amount of housing. This is from (1.29)
3. The population density is equal to minus the ratio of the land price gradient to the marginal cost of commuting. This comes from quantitative prediction 1 plus equation (1.31).
4. As we move away from the center, the decrease in rent is proportional to commuting costs scaled by lot size. This is from prediction 1 and equation (1.19). So the difference between the rent at the center and the rent at the border should be these accumulated rent reductions scaled by populations. As Duranton and Puga (2015) argue, this is problematic to test because of the featureless plane assumption.